A Study on Non-Linear Eigenvalue Problems for Waveguide-Coupled Electromagnetic Cavities

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The frequency domain simulation of electromagnetic resonators with finite methods (Finite Elements, Finite Integration) leads to large-scale eigenvalue formulations. Any kind of loss mechanism, such as conduction or radiation through open ports of the structure, necessarily corresponds to complex eigenvalues, where the ratio of the real and imaginary parts of the eigenvalues defines the (internal and/or external) *Q*-values of the modes. In the case of an external coupling to waveguides with dispersion characteristics the eigenvalue formulation becomes non-linear, i.e. the system matrix depends on the eigenfrequency to be computed. We use an integral solver for such non-linear eigenvalue problems and perform a study on some varying formulations. Some important properties are discussed such as the spectral properties, the influence of some approximations therein, as well as the efficiency of the overall solution process.

Index Terms—**Eigenvalues and Eigenfunctions, Finite Difference Methods, Numerical Simulation.**

I. Introduction

The calculation of eigenvalues in electromagnetic cavities is a standard task in many simulation tools, and a variety of efficient solvers are available. However, the algebraic situation becomes much more difficult if the cavities are externally coupled to waveguides with non-TEM modes, since the dispersion characteristics of such waveguide modes introduces an additional expression in the system matrix which depends on the frequency. Hence, the resulting formulation becomes non-linear w.r.t. the eigenvalue (the resonance frequency) and requires a careful choice of a suitable matrix formulation and solution process. We mainly use the notation of the finite integration technique (FIT) here, but most results are valid also for related approaches such as finite elements (FE).

II. Formulations

A. Finite Integration Technique (FIT)

The degrees of freedom of FIT [1] are the so-called integral state variables: electric and magnetic voltages $\widehat{e}_i, \widehat{h}_j$. They are defined on the edges $L_i, \tilde{L}_i$ of the primary grid G and the dual grid $\tilde{G}$, respectively:

$$\widehat{e}_i = \int_{L_i} \vec{E} \cdot \mathrm{d}\vec{s}\,, \qquad \widehat{h}_i = \int_{\tilde{L}_i} \vec{H} \cdot \mathrm{d}\vec{s}\,. \qquad (1)$$

Combining them to algebraic vectors, Maxwell's grid equations in frequency domain can be written as (without currents)

$$\mathbf{C}\widehat{\mathbf{e}} = -j\omega \mathbf{M}_\mu \widehat{\mathbf{h}}, \qquad \mathbf{C}^T \widehat{\mathbf{h}} = j\omega \mathbf{M}_\varepsilon \widehat{\mathbf{e}}\,, \qquad (3)$$

with the curl-operator $\mathbf{C}$, and the linear material matrices $\mathbf{M}_\varepsilon$ and $\mathbf{M}_\mu$. Eliminating the magnetic voltages, we obtain the standard eigenvalue equation

$$(\mathbf{C}^T \mathbf{M}_\mu^{-1} \mathbf{C} - \omega^2 \mathbf{M}_\varepsilon)\ \widehat{\mathbf{e}} = 0\,. \qquad (5)$$

B. Non-linear eigenvalue problem

The first *non-linear eigenvalue formulation* for waveguide-coupled cavity models is adapted from [2] (for finite elements, a similar formulation is reported in [3]). The following steps of an extended state-space formulation are required:

- Define a coupling matrix **B** which contains columnwise the fields of the port modes, properly normalized at a reference frequency ω_{0i}. This matrix can be used for both an excitation of the system by generalized currents at the ports, and the extraction of generalized voltages, see, e.g., [4].
- A proper relation between generalized currents and voltages (their ratio equals the line impedance) provides the requested situation where there are no incoming waves at all ports. This defines the system matrix for the eigenvalue problem.

After some manipulations we obtain the following formulation which is non-linear in the eigenvalue ω :

$$\mathbf{T}(\omega)\,\widehat{\mathbf{e}} = \left(\mathbf{C}^T \mathbf{M}_\mu^{-1} \mathbf{C} - \omega^2 \mathbf{M}_\varepsilon + j\omega \mathbf{B}\mathbf{P}(\omega)\mathbf{B}^T\right)\widehat{\mathbf{e}} = \mathbf{0}\,. \qquad (6)$$

The expression $\mathbf{B}\mathbf{P}(\omega)\mathbf{B}^T$ models the radiation through external waveguides. Since the coupling matrix **B** refers to a normalization of the modes at reference frequencies ω_{0i}, the diagonal scaling matrix **P** is required which contains normalization coefficients for the generalized impedances $Z(\omega_{0i})\,/\,Z(\omega)$ [2][4]. There is one column in **B** and one entry in **P** for each mode in one of the ports. For TE and TM modes the entries of **P** read

$$p_{i,i}(\omega) = \begin{cases} \gamma_i & (TE) \\ \gamma_i^{-1} & (TM) \end{cases}, \qquad \gamma_i = \frac{\sqrt{1-(\omega_{c,i}/\omega)^2}}{\sqrt{1-(\omega_{c,i}/\omega_{0,i})^2}}\,. \qquad (7)$$

Note that this formulation is more or less identical to the FE formulation reported in [2], only the parts representing the stiffness and mass matrices have to be exchanged when finite elements are replaced by finite integration.

C. Alternative Formulations

A simple way to avoid a non-linear eigenvalue solver is to linearize the formulation: The matrix in (6) is evaluated at some initial guess, $\mathbf{T}(\omega^{(0)})$, allowing to apply a simple linear eigenvalue solver, augmented by some steps of a fixed-point iteration (with updates $\omega^{(k)}$ of the evaluation frequency). Comparing the overall effort it should be noted that this procedure is valid only for a single eigensolution and may have to be repeated accordingly. Additionally, it may be hard to find an appropriate and sufficiently accurate starting value $\omega^{(0)}$, and some existing eigenvalues may not be found at all. Nevertheless, this linearization approach can be used for validation purposes.

An alternative formulation follows directly the commonly used time domain implementation for scattering parameter simulations. Instead of generalized voltages and currents, defined directly in the port planes, we extract the wave amplitudes along the coupled waveguide in two different slices of a Cartesian mesh from the field vector. Using the dispersion characteristics $k_z(\omega)$ of each mode these amplitudes are used to realize an absorbing boundary condition for waveguide modes in frequency domain. Proposed in [5] for excited systems (including an extended right hand side vector), the resulting matrix can also be used for the eigenvalue problem, simply setting the amplitudes of all incoming waves to zero. Note that although the same radiating modes (complex eigenvalues) are supported, this formulation features a different non-linearity w.r.t. frequency, including an expression like $\exp(-j\, k_z(\omega)\, \Delta z)$.

Finally, the standard time domain solver itself can be used to produce complex eigenfrequencies for the "open" problem: The structure is excited by an artificial internal current source, which has to be appropriately located to excite the searched mode fields, and which should exhibit a suitable bandwidth. The resulting, decaying time signals (e.g., directly at the ports) can be used to extract resonance frequencies and Q values. This well-established approach can be performed using standard time domain solvers, e.g. in commercial tools, and an experienced user will probably find all desired eigensolutions. However, the simulation time for accurate results can be large for high-Q resonances (with long transients and settling times), and there is some extra effort to obtain also the eigenvectors (the fields) if required. Further on, there will be a small deviation to the frequency domain results due to the additional numerical dispersion effects in time domain.

III. Integral Solver

The outline of the integral solver [6] has been presented in [7] and will not be repeated here. The main idea is to calculate an approximation of a line integral along a closed contour in the complex ω-plane, where at each evaluation point a full-size matrix has to be inverted. The overall numerical effort is very high and crucially depends on many implementation issues.

IV. Numerical Results

The numerical results show that all formulations are capable to find complex eigenvalues of a simple test structure (rectangular waveguide with dielectric inset, see [7]). The deviation of the computed eigenvalues to a semi-analytical reference solution is below any practical needs (and thus not evaluated here in detail), proving their validity.

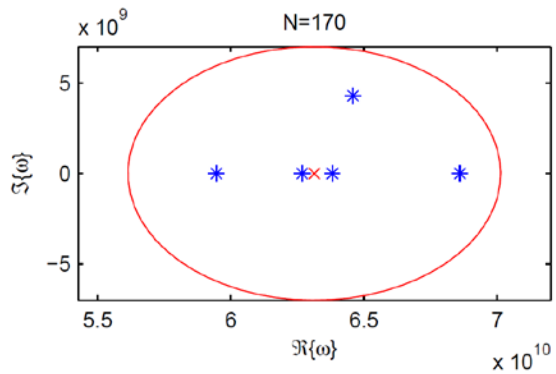

Fig. 1. Eigenvalues in the complex plane (blue), found by the integral solver within a contour around some midpoint (red).

A major advantage of the integral solver is the fact that it guarantees to find the *complete* set of all eigenvalues (and corresponding fields) within a given contour (Fig. 1), however at comparatively high numerical cost. For the linearized fixed-point iteration as well as for the time domain approach each mode has to be searched for separately, and both require some a-priori knowledge of these radiating modes. The magnitude of their Q values, as a measure of the deviation of the fields in the closed and the open structure, has some impact here. More details will be given in the presentation.

V. Acknowledgement

The implementation of the solver is due to B. Bandlow [7].

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